

PERMUTATION SIGN UNDER THE ROBINSON-SCHENSTED-KNUTH CORRESPONDENCE

Astrid Reifegerste

Institut für Mathematik, Universität Hannover

Welfengarten 1, 30167 Hannover, Germany

reifegerste@math.uni-hannover.de

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ABSTRACT. We show how the sign of a permutation can be deduced from the tableaux induced by the permutation under the Robinson-Schensted-Knuth correspondence. The result yields a simple proof of a conjecture on the squares of imbalances raised by Stanley.

1 INTRODUCTION

The Robinson-Schensted-Knuth correspondence is a classical bijection between permutations and pairs of standard Young tableaux of the same shape. It was given in [6], and followed by numerous papers studying its combinatorial properties.

There are some well-known relations between certain permutation statistics on the one hand and tableaux statistics on the other hand. Schensted's classical Theorem [6] states that the length of the longest increasing and decreasing subsequence in a permutation equals the length of the first row and column of the associated tableaux, respectively. Schützenberger [7] showed that the descent set of a permutation and the descent set of its recording tableau coincide.

In [4], the question of how to obtain the sign from the associated pair of tableaux was answered for 321-avoiding permutations, that is, for permutations having no decreasing subsequence of length three. The problem was motivated by the wish for refining the sign-balance property of 321-avoiding permutations respecting the length of the longest increasing subsequence.

In this paper, we will give a general answer.

It is known for a long time how the sign of an involution can be deduced from its corresponding tableau. A theorem of Schützenberger [8] says that the number of fixed points of an involution is equal to the number of columns of odd length in the associated tableau. Consequently, the sign is determined by the total length of all even-indexed tableau rows. In [1], Beissinger described an elegant algorithm for constructing the tableau in bijection with an involution. Her bijective

proof of Schützenberger’s result makes use of the transparency of her algorithm with regard to the cycle structure.

Due to Knuth’s equivalence relation, we can extend the case of involutions, treated in Section 3, to arbitrary permutations. To this end, we only have to consider the effect having an elementary Knuth transformation on the recording tableau. In Section 4, the resulting observation will be described, and the main result will be proved. Therefore, the sign of a permutation is the product of the signs of the associated tableaux and the parity of the total length of the even-indexed rows.

In Section 5, we apply this result to prove Stanley’s conjecture [11] on the squares of imbalances. Furthermore we use the explicit knowledge of the imbalance of hooks for determining the joint distribution of the sign and the length of the longest increasing subsequence for a special kind of pattern-avoiding permutations.

2 DEFINITIONS AND NOTATIONS

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_l > 0$ be a partition of n , denoted by $\lambda \vdash n$. We use this notation for both the partition and its corresponding Young diagram. A standard Young tableau (or just tableau in the following) of shape λ is a labeling of the squares of λ with the numbers $1, 2, \dots, n$ so that the rows and columns are increasing.

We assume that the reader is familiar with the combinatorics of Young tableaux, and, especially, with the Robinson-Schensted-Knuth correspondence (for details see, e.g., [2] or [10]).

This correspondence gives a bijection between permutations of the symmetric group \mathcal{S}_n and pairs (P, Q) of tableaux of the same shape $\lambda \vdash n$. Here P is called the *insertion tableau* and Q the *recording tableau* of the permutation. If necessary, we will denote the tableaux that are obtained at the i th step of the algorithm by P_i and Q_i . (By definition, we have $P = P_n$ and $Q = Q_n$.)

For a tableau T , we call a pair (i, j) of entries an *inversion of T* if $j < i$ and j is contained in a row below the row of i . We write $\text{inv}(T)$ to denote the number of inversions of T . Furthermore, we define the *sign of a tableau T* as

$$\text{sign}(T) = (-1)^{\text{inv}(T)}.$$

Any tableau T can be identified with its *row word*, denoted by $w(T)$, which is obtained by reading the entries row by row from left to right, starting from the top. (Note that the inversions of a tableau are just the inversions of this word.)

3 THE CASE OF INVOLUTIONS – BEISSINGER’S ALGORITHM

An important property of the Robinson-Schensted-Knuth correspondence is its symmetry: the recording tableau for a permutation π is just the insertion tableau for the inverse permutation π^{-1} . This statement has already appeared in the work of Robinson [5]; the first proof was given by Schützenberger [7]. Later, the result was demonstrated again by Viennot [12] using the symmetry of his geometric construction. Thus each involution $\pi \in \mathcal{S}_n$ is associated with a pair (P, P) where P is a tableau of shape $\lambda \vdash n$. Consequently, the correspondence gives a bijection between involutions and tableaux.

The question of how to see the sign of an involution by looking at its corresponding tableau can be easily answered using a simple algorithm given by Beissinger [1]. Her construction yields the tableau in bijection with an involution, and works as follows.

Given an involution $\pi \in \mathcal{S}_n$, write π as a product of disjoint cycles c_1, c_2, \dots, c_k in increasing order of their greatest element and with $a < b$ for each cycle (a, b) . Starting with the empty tableau, P is obtained recursively by applying the following procedure, for $i = 1, \dots, k$:

- If $c_i = (a)$, then place a at the end of the first row of P .
- If $c_i = (a, b)$, then insert a into P . Let l be the index of the row in which the insertion process stops. Place b at the end of the row $l + 1$.

One consequence of the algorithm is a direct proof of a result of Schützenberger [8, p. 93] which we give here in the following equivalent formulation.

Theorem 3.1 *Let $\pi \in \mathcal{S}_n$ be an involution. Then $\text{sign}(\pi) = (-1)^e$ where e denotes the total length of all even-indexed rows of the corresponding tableau.*

Proof. By description of the algorithm, inserting the elements of a 2-cycle creates two squares in consecutive rows. Hence the number of 2-cycles of π equals the number of squares in all even-indexed rows of P . This yields the assertion. \square

Example 3.2 Let $\pi = 4\ 8\ 7\ 1\ 9\ 6\ 3\ 2\ 5 = (1, 4)(6)(3, 7)(2, 8)(5, 9) \in \mathcal{S}_9$. Beissinger’s algorithm generates the corresponding tableau in five steps:

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Its even-indexed rows consist of 4 squares in all. Thus π is even.

Remark 3.3 Another proof of Theorem 3.1 is given in [10, Exercise 7.28a]. It is based on growth diagrams of permutations.

4 THE GENERAL CASE – KNUTH EQUIVALENCE

Now we extend the result given for involutions in the previous section to arbitrary permutations. To this end, another classical property of the Robinson-Schensted-Knuth correspondence will be used. In [3], Knuth gave a characterization of permutations having the same insertion tableau.

An *elementary Knuth transformation* of a permutation is the application of one of the transformations

$$acb \mapsto cab, \quad cab \mapsto acb, \quad bac \mapsto bca, \quad bca \mapsto bac \quad \text{where } a < b < c$$

to three consecutive letters of the permutation. Two permutations are called *Knuth-equivalent* if they can be changed into each other by a sequence of elementary Knuth transformations.

The result of Knuth [3, Theorem 6] is the following: permutations are Knuth-equivalent if and only if their insertion tableaux coincide. Because of the symmetry of the Robinson-Schensted-Knuth correspondence, this also implies a characterization of permutations having the same recording tableau.

The proof of the generalization of Theorem 3.1 bases on the following simple observation.

Lemma 4.1 *Any elementary Knuth transformation reverses the sign of the recording tableau.*

Proof. Let $\pi \in \mathcal{S}_n$ be a permutation and (P, Q) the corresponding pair of tableaux. We show that the recording tableau Q' of a permutation σ which is obtained from π by an elementary Knuth transformation arises from Q by exchanging two consecutive numbers.

By symmetry, Q and Q' are the insertion tableaux of $p = \pi^{-1}$ and $s = \sigma^{-1}$, respectively. First, let

$$s = p_1 \cdots p_{a-1} \, k+2 \, p_{a+1} \cdots p_{b-1} \, k \, p_{b+1} \cdots p_{c-1} \, k+1 \, p_{c+1} \cdots p_n = (k+1, k+2) \cdot p$$

where $a < b < c$. Then σ has arose from π by the transformation $bac \mapsto bca$. Clearly, we have $Q_i = Q'_i$ for $i = 1, \dots, a-1$. Of course, the smallest integer in the first row of Q_{a-1} that is larger than $k+1$ is just as the smallest integer that is larger than $k+2$. Thus the tableaux $Q_a = Q_{a-1} \leftarrow k+1$ and $Q'_a = Q_{a-1} \leftarrow k+2$ are of the same shape. (In the following, we use the notation $T \leftarrow i$ as well for the tableau resulting from the row insertion of the integer i into the tableau T as for the insertion process itself.) More exactly, Q_a and Q'_a only differ in the entry of a single square (in the first row) which is occupied by $k+1$ in Q_a and by $k+2$ in Q'_a . Since $p_i \notin \{k+1, k+2\}$ for $i = a+1, \dots, c-1$, the insertion $Q_{i-1} \leftarrow p_i$ takes the same path

as $Q'_{i-1} \leftarrow p_i$. (If $k+1$ is the smallest integer in Q_{i-1} which is larger than an element p_i , then $k+2$ is it as well in Q'_{i-1} .) In this way, we obtain the tableaux

$$Q_{c-1} = \begin{array}{|c|} \hline \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k+1 \end{array} \\ \hline \end{array} \quad Q'_{c-1} = \begin{array}{|c|} \hline \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k+2 \end{array} \\ \hline \end{array}$$

which are identical, except from the entry of the gray square. Note that this square cannot belong to the first row since $k+1$ and $k+2$, respectively, have been bumped at the latest when inserting $p_b = k$. Furthermore, the entry k is placed above $k+1$ and $k+2$, respectively. If k is bumped into the row containing $k+1$ or $k+2$, then it for its part bumps these numbers. Therefore, the insertions $Q_{c-1} \leftarrow k+2$ and $Q'_{c-1} \leftarrow k+1$ run identically. Finally, inserting the remaining elements p_{c+1}, \dots, p_n successively yields the tableaux Q and Q' which can be transformed into each other by changing the elements $k+1$ and $k+2$:

$$Q = \begin{array}{|c|} \hline \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k+2 \end{array} \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k+1 \end{array} \\ \hline \end{array} \quad Q' = \begin{array}{|c|} \hline \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k+1 \end{array} \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k+2 \end{array} \\ \hline \end{array}$$

Why? Since k has been inserted before $k+2$ in Q and before $k+1$ in Q' , it is not possible that $k+2$ and $k+1$ are bumped into the row which contains $k+1$ and $k+2$, respectively. When inserting p_{c+1}, \dots, p_n successively into Q_c , the element $k+2$ can be bumped into the row containing k at most. Then it occupies the square directly to the right of that of k . By construction, $k+2$ does not move while it is contained in the same row as k . Analogously, $k+1$ cannot pass by k in Q' . Consequently, we have $\text{inv}(Q) = \text{inv}(Q') + 1$.

Consider now the second kind of an elementary Knuth transformation: $acb \mapsto cab$. Let

$$s = p_1 \cdots p_{a-1} k+1 p_{a+1} \cdots p_{b-1} k+2 p_{b+1} \cdots p_{c-1} k p_{c+1} \cdots p_n = (k, k+1) \cdot p$$

for some $a < b < c$. By reasoning similarly as above, we can show that

$$Q_c = \begin{array}{|c|} \hline \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k+1 \end{array} \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k+2 \end{array} \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k \end{array} \\ \hline \end{array} \quad Q'_c = \begin{array}{|c|} \hline \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k \end{array} \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k+2 \end{array} \\ \hline \begin{array}{c} \blacksquare \\ \nearrow k+1 \end{array} \\ \hline \end{array}$$

All the numbers $i \neq k, k+1$ are placed at the same position in Q'_c as they have in Q_c . Note that $k+2$ can be bumped into the row of k in Q_c and $k+1$ in Q'_c at most, respectively; in this case, the numbers occupy adjacent squares. Let us consider what happens when we insert the

remaining elements into Q_c and Q'_c . If $k+2$ is above k in Q_{i-1} and $k+1$ in Q'_{i-1} , respectively, then $k+1$ bumps $k+2$ during the insertion $Q_{i-1} \leftarrow p_i$ ($i > c$) if and only if k bumps $k+2$ during $Q'_{i-1} \leftarrow p_i$. Thus we have $\text{inv}(Q) = \text{inv}(Q') + 1$ if the numbers k , $k+1$, and $k+2$ have the relative positions in Q and Q' as shown in the above figure. Assume now that $k+2$ has been bumped into the row of k and $k+1$, respectively:



If $k+1$ and k is bumped into this row by a sequence of insertions, respectively, then we obtain



Clearly, the bumped numbers occur at the same position in the next row since they are consecutive. Each further insertion will take the same path through Q_j as through Q'_j . Note again that $k+1$ and $k+2$, respectively, does not move while k is situated to their left. Thus $k+1$ and $k+2$ are contained in different rows, and we have $\text{inv}(Q) = \text{inv}(Q') - 1$. \square

Remark 4.2 Obviously, the inversion numbers of permutations which are connected by an elementary Knuth relation differ by 1. Since a transformation $bac(b) \mapsto bca(b)$ can cause as well the increase as the decrease (by 1) of the inversion number of the recording tableau it seems that the problem of recovering the inversion number from the tableaux is unlike more difficult.

Theorem 4.3 *Let $\pi \in \mathcal{S}_n$ be a permutation and (P, Q) its associated pair of tableaux. Then*

$$\text{sign}(\pi) = \text{sign}(P) \cdot \text{sign}(Q) \cdot (-1)^e$$

where e is the total length of all even-indexed rows of P .

Proof. Consider two special elements of the Knuth class containing π : the involution σ , and the permutation τ whose recording tableau, denoted by I , has no inversions. (Note that the row word of I is the identity in \mathcal{S}_n .)

$$\begin{array}{ccccc}
\sigma & \longrightarrow & \tau & \longrightarrow & \pi \\
\updownarrow & & \updownarrow & & \updownarrow \\
(P, P) & & (P, I) & & (P, Q)
\end{array}$$

Obviously, the tableau P can be transformed into I by $\text{inv}(P)$ exchanges of consecutive elements i and $i+1$ (necessarily, contained in different rows). In terms of row words, each of these

transformations means the multiplication of $w(P)$ with an adjacent transposition $(i, i + 1)$ in \mathcal{S}_n . Analogously, Q arises from I by $\text{inv}(Q)$ exchanges of consecutive elements. Therefore, the assertion follows immediately from Theorem 3.1 and Lemma 4.1. \square

Example 4.4 Consider the permutation $\pi = 2\ 9\ 1\ 5\ 6\ 4\ 8\ 3\ 7 \in \mathcal{S}_9$. It corresponds to the pair

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 7 \\ \hline 2 & 4 & 8 & \\ \hline 5 & & & \\ \hline 9 & & & \\ \hline \end{array} \quad Q = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 & 9 & \\ \hline 6 & & & \\ \hline 8 & & & \\ \hline \end{array}$$

of tableaux. We have $\text{inv}(P) = 8$ and $\text{inv}(Q) = 7$. The number of squares laying in the second or fourth row of P equals 4. Thus π is an odd permutation.

Remark 4.5 In [4], the problem of how to obtain the sign from the associated tableaux was solved for 321-avoiding permutations. Note that the tableaux have at most two rows in this case. Let $\pi \in \mathcal{S}_n$ be a 321-avoiding permutation and (P, Q) its associated pair of tableaux. [4, Proposition 2.1] states that $\text{sign}(\pi) = (-1)^{s+e}$ where s denotes the sum of all entries of the second row of P and Q and e is the length of this row.

In general, s is not equal to $\text{inv}(P) + \text{inv}(Q)$. For example, $\pi = 2\ 5\ 1\ 6\ 8\ 3\ 9\ 4\ 7 \in \mathcal{S}_9$ is in bijection with the pair

$$P = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 7 & 9 \\ \hline 2 & 5 & 6 & 8 & \\ \hline \end{array} \quad Q = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 7 \\ \hline 3 & 6 & 8 & 9 & \\ \hline \end{array}$$

of tableaux for which $s = 47$. On the other hand, $\text{inv}(P) = 9$ and $\text{inv}(Q) = 4$.

Alternatively to the proof given in [4], we can derive the result from Theorem 4.3. Let i_k be the entry occupying in the k th square of the first row of P . Then $i_k - k$ of the elements contained in the second row are smaller than i_k . Similarly, the entry j_k of the k th square in the first row of Q causes $j_k - k$ inversions. Since $s = n(n + 1) - (i_1 + \dots + i_{n-e} + j_1 + \dots + j_{n-e})$, we have

$$\text{inv}(P) + \text{inv}(Q) = \sum_{k=1}^{n-e} (i_k - k) + \sum_{k=1}^{n-e} (j_k - k) \equiv s \pmod{2}.$$

5 CONSEQUENCES

Recently, Stanley [11] considered sign-balanced posets and gave particularly some new results for posets arising from partitions. (For an introduction to the theory of posets see [10].)

For a partition λ , let

$$I_\lambda(q) = \sum_T q^{\text{inv}(T)}$$

be the generating function for the tableaux of shape λ by the number of inversions. The integer $I_\lambda = I_\lambda(-1)$ is called the *imbalance* of λ .

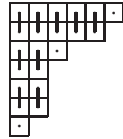
In this section, we discuss some consequences of the relation between the sign of a permutation and the signs of its associated tableaux. On the one hand, we give a simple proof for one of Stanley's results on imbalances of partitions. On the other hand, we can use the knowledge about imbalances of special shapes for refining the enumeration of certain restricted permutations concerning their sign and the length of their longest increasing subsequence.

There are as many even permutations in \mathcal{S}_n as odd ones. Applying Theorem 4.3, we can interpret this well-known fact in terms of tableaux. This extends one of Stanley's results and proves a special case of a conjecture he has given. By [11, Theorem 3.2b], we have

$$\sum_{\lambda \vdash 2m} (-1)^{v(\lambda)} I_\lambda^2 = 0$$

where $v(\lambda)$ denotes the maximum number of disjoint vertical dominos that fit in the shape λ . The proof uses a bijection between colored biwords and pairs of standard domino tableaux of the same shape which was established by Shimozono and White [9, Theorem 30].

Note that $v(\lambda)$ just counts the number of squares contained in an even-indexed row of λ :



To emphasize the equivalent definition, we rename the statistic v and write $e(\lambda)$ to denote the sum $\lambda_2 + \lambda_4 + \dots$ of all even-indexed parts of λ .

The case $t = 1$ of Conjecture [11, 3.3b] claims that the above equation is correct for partitions of an arbitrary integer n . Theorem 4.3 yields the proof.

Theorem 5.1 *For all $n \geq 1$, we have*

$$\sum_{\lambda \vdash n} (-1)^{e(\lambda)} I_\lambda^2 = 0.$$

Proof. From the sign-balance on \mathcal{S}_n we obtain

$$0 = \sum_{\pi \in \mathcal{S}_n} \text{sign}(\pi) = \sum_{\lambda \vdash n} \sum_{(P, Q)} (-1)^{e(\lambda)} \text{sign}(P) \text{sign}(Q) = \sum_{\lambda \vdash n} (-1)^{e(\lambda)} I_\lambda^2$$

where (P, Q) ranges over all pairs of tableaux of shape λ . □

For certain shapes λ (hooks and rectangles), the imbalance I_λ has been determined explicitly. The characterization of permutations whose associated tableaux have exactly such a shape allows a refined enumeration of these permutations regarding their sign and even the length of

their longest increasing subsequence. These considerations lead to pattern-avoiding permutations. Given a permutation $\pi \in \mathcal{S}_n$ and a permutation $\tau \in \mathcal{S}_k$, we say that π *avoids the pattern* τ if there is no sequence $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that the elements $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_k}$ are in the same relative order as $\tau_1, \tau_2, \dots, \tau_k$.

In [4, Theorem 1.1], the joint distribution of **sign** and **lis**, the length of the longest increasing subsequence, was given for 321-avoiding permutations. As an application of result [11, Proposition 3.4] which deals with the weighted imbalances of hooks, we obtain the number of even and odd permutations which avoid both 213 and 231 respecting the statistic **lis**, respectively.

Proposition 5.2 *Let $\pi \in \mathcal{S}_n$ be a permutation and λ the shape of its insertion tableau P . Then π avoids 213 and 231 if and only if $w(P) = 12 \cdots n$ and $\lambda = (k, 1^{n-k})$ for $k \in \{1, \dots, n\}$.*

Proof. First we show that permutations having insertion tableaux of this very special form can be characterized as follows. Let $i_1 < i_2 < \dots < i_d$ be the descents of $\pi \in \mathcal{S}_n$ and $j_1 < j_2 < \dots < j_{n-d}$ the remaining positions. (An integer $i \in \{1, \dots, n-1\}$ is called a *descent* of π if $\pi_i > \pi_{i+1}$.) Then the insertion tableau P of π has no inversions and is of shape $(k, 1^{n-k})$ if and only if $d = n - k$ and

$$\pi_{i_1} \pi_{i_2} \cdots \pi_{i_d} = n(n-1) \cdots (k+1) \quad \text{and} \quad \pi_{j_1} \pi_{j_2} \cdots \pi_{j_{n-d}} = 12 \cdots k.$$

Suppose that

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & k \\ \hline l & & & \\ \hline \vdots & & & \\ \hline n & & & \\ \hline \end{array}$$

where $l = k + 1$. By reversing the steps in the Robinson-Schensted-Knuth algorithm, we obtain the elements $\pi_n, \pi_{n-1}, \dots, \pi_1$ from the entry contained in the rightmost square of the first row of the tableaux $P = P_n, P_{n-1}, \dots, P_1$. To see this note that the shape of the recording tableau Q is a hook as well. Clearly, if the element i appears at the end of the first row in the subtableau Q_i , then we find π_i at the same position in P_i . Otherwise, if the element i occupies the bottom square in the first column of Q_i , then by applying the reverse row-insertion to P_i with that square, the rightmost element is bumped out of the first row as well since all entries of the hook leg are larger than the entries of the hook arm. Therefore, a descent arises whenever an element is bumped out that was not placed in the first row of P . Since the reverse procedure builds up the permutation from right to left, the descent tops (that is, the elements $\pi_{i_1}, \dots, \pi_{i_{n-k}}$) are decreasing while the remaining letters form the sequence $1, 2, \dots, k$.

Conversely, it is obvious that a permutation with this descent structure produces an insertion tableau whose row word is the identity and which has the shape of a hook.

But these permutations are exactly the $\{213, 231\}$ -avoiding ones. If $\pi \in \mathcal{S}_n$ contains no pattern

213 or 231, then for any descent i there exists no integer $j > i$ for which $\pi_j > \pi_i$ (otherwise we have the pattern 213 in π), and for any non-descent i there exists no $j > i$ with $\pi_j < \pi_i$ (otherwise the pattern 231 occurs). Consequently, the descent tops of π have to be decreasing, and moreover, equal to $n, n-1, \dots, k+1$; the remaining elements are arranged in increasing order. By similar reasoning, we obtain the converse. \square

Corollary 5.3 *Let A_n denote the set of permutations in \mathcal{S}_n which avoid both 213 and 231. For all $n \geq 1$, we have*

$$\sum_{\pi \in A_n} \text{sign}(\pi) q^{\text{lis}(\pi)} = q(q+1)^{\lfloor (n-1)/2 \rfloor} (q-1)^{\lfloor n/2 \rfloor}.$$

In particular, there are as many even as odd permutations in A_n .

Proof. Let $\pi \in A_n$ with $\text{lis}(\pi) = k$. By Proposition 5.2, the tableaux in bijection with π are of shape $\lambda = (k, 1^{n-k})$. (Recall that the length of the longest increasing subsequence equals the length of the first row of the associated tableaux.) Clearly, $e(\lambda) = \lceil \frac{n-k}{2} \rceil$ and $e(\lambda') = \lfloor \frac{k}{2} \rfloor$ for the conjugate partition λ' . By the proof of [11, Proposition 4.3], we have $I_\lambda = 0$ if n is odd and k even and $I_\lambda = \binom{h(n)}{h(k)}$ otherwise where $h(x) = \lfloor (x-1)/2 \rfloor$. By means of Theorem 4.3, we can express the coefficients of the generating function

$$F(q) = \sum_{\pi \in A_n} \text{sign}(\pi) q^{\text{lis}(\pi)}$$

in terms of imbalances:

$$F(q) = \sum_{k=1}^n \sum_Q (-1)^{e(\lambda)} \text{sign}(Q) q^k = \sum_{k \text{ odd}} q^{2e(\lambda')+1} (-1)^{e(\lambda)} I_\lambda + \sum_{k \text{ even}} q^{2e(\lambda')} (-1)^{e(\lambda)} I_\lambda$$

where Q ranges over all tableaux of shape $\lambda = (k, 1^{n-k})$. If n is odd, then I_λ vanishes whenever k is even. Thus

$$F(q) = q \sum_{k=1}^n q^{2e(\lambda')} (-1)^{e(\lambda)} I_\lambda = q(q^2 - 1)^{\frac{n-1}{2}}.$$

In case of even n , we have

$$\sum_{k \text{ even}} q^{2e(\lambda')} (-1)^{e(\lambda)} I_\lambda = \sum_{k \text{ even}} q^k (-1)^{\frac{n-k}{2}} \binom{\frac{n-2}{2}}{\frac{k-2}{2}} = -q \sum_{k \text{ odd}} q^{2e(\lambda')+1} (-1)^{e(\lambda)} I_\lambda$$

and hence

$$F(q) = q(1-q) \sum_{k \text{ odd}} q^{2e(\lambda')} (-1)^{e(\lambda)} I_\lambda = q(q-1)(q^2-1)^{\frac{n}{2}-1}.$$

\square

Remark 5.4 Proposition 5.2 gives a bijection between $\{213, 231\}$ -avoiding permutations and standard Young tableaux of hook shape (consider only the recording tableaux). Therefore, it also yields a proof – although not the most obvious one – for the known fact $|A_n| = 2^{n-1}$. Because of the symmetry of the Robinson-Schensted-Knuth correspondence, we may replace the pattern 231 in Corollary 5.3 with 312.

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